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Two generalizations of column-convex polygons

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Abstract

Column-convex polygons were first counted by area several decades ago, and the result was found to be a simple, rational, generating function. In this work we generalize that result. Let a p-column polyomino be a polyomino whose columns can have $1, 2, \ldots, p$ connected components. Then column-convex polygons are equivalent to 1-convex polyominoes. The area generating function of even the simplest generalization, namely 2-column polyominoes, is unlikely to be solvable. We therefore define two classes of polyominoes which interpolate between column-convex polygons and 2-column polyominoes. We derive the area generating functions of those two classes, using extensions of existing algorithms. The growth constants of both classes are greater than the growth constant of column-convex polyominoes. Rather tight lower bounds on the growth constants complement a comprehensive asymptotic analysis.

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1. Introduction

The enumeration of polyominoes is a topic of great interest to chemists, physicists and combinatorialists alike [20]. In chemical terms, any polyomino (with hexagonal cells) is a possible benzenoid hydrocarbon. In combinatorics, polyominoes are of interest in their own right because several polyomino models have mathematically appealing exact solutions. Furthermore, they are also relevant to various problems of tilings [12]. Polyominoes are extensively studied, in one form or another, in a wide variety of problems of great interest to physicists. In particular, we note their investigation under the name *lattice animals*, in the study of percolation [15, 17], in the graphical representation of the Ising model and its extension to the Potts model, and in the study of the properties of branched polymers [14, 21, 23].

They are also a representative of a class of problems that appear to be unsolvable—notably the enumeration, by area or perimeter, of self-avoiding polygons, polyominoes and other classes of graphs that are relevant to the Ising model and to percolation. The principal line of attack on such problems is to simplify them until they are solvable, in the hope that the essential physics is not destroyed in the process. That is the approach taken in this paper, where the models proposed, while still solvable, are closer to the ultimate problem of full polyomino enumeration than has previously been attained. Further, by use of Bousquet–Mélou's [3] and Svrtan's [10] upgraded version of the Temperley methodology [22], we give the solution of two problems previously out of reach due to their complexity. These developments may spur further advances in this class of problem.

One frequently cited polyomino model is that of column-convex polygons[3]. We will consider two versions of column-convex polygons: the first composed of square cells and the second of hexagonal cells. Both versions have a rational area generating function. For the version with square cells, the area generating function was found independently by Pólya [19] in 1938 *or* 1969 and by Temperley [22] in 1956. That was perhaps the earliest major result in polyomino enumeration. For the version with hexagonal cells, the area generating function was found by Klarner in 1967 [18]. The growth constant of square-celled column-convex polygons is $\mu = 3.205\,569\ldots$, while the growth constant of hexagonal-celled column-convex polygons is $\mu = 3.863\,130\ldots$. (By the growth constant we mean the limit $\lim_{n\to\infty} \sqrt[n]{a_n}$, where a_n denotes the number of n-celled elements in a given set of polyominoes.) In both cases the area generating function is a simple pole, so that $a_n \sim \text{const} \times \lambda^n$.

There exist some models which are supersets of column-convex polygons and are still solvable. These models are called *m-convex polygons* [16], *prudent polygons* [11], *cheesy polyominoes* [6], *polyominoes with cheesy blocks* [7], *column-subconvex polyominoes* [9] and *simple-2-column polyominoes* [8]. The former two models can be enumerated by perimeter and area, whereas the latter four models have been enumerated only by area.

In this paper, we present two models: column-subconvex polyominoes and simple-p-column polyominoes. In a column-subconvex polyomino, a column may have one or two connected components. However, the gap within a two-component column must not be greater than m cells in size, where m is a positive integer which we fix in advance. (If there were no other requirements besides 'a column may have one or two connected components', the model would still be too hard, i.e. not amenable to exact enumeration.) In a simple-p-column polyomino, a column may have $1, 2, \ldots, p$ connected components. The gaps within a column can be of any size. However, columns with more than one connected components must not be adjacent to one another. In this paper we discuss the simplest version, simple-2-column polyominoes.

Column-subconvex polyominoes are somewhat easier to deal with when cells are hexagons than when cells are squares. Thus, we computed the area generating function for $m = 1$ column-subconvex polyominoes with hexagonal cells and for simple-2-column polyominoes with square cells. Both of these generating functions are complicated q-series. As mentioned above, we made use of Bousquet–Mélou's [3] and Svrtan's [10] upgraded version of Temperley's methodology [22].

The computations are rather long and intricate. Therefore, in this paper we only give an outline of the proofs, though with enough detail that the methods may be applied by others to new problems.

[3] We distinguish between polygons and polyominoes in that the former cannot have internal holes. As a consequence, the *perimeter* generating function for polygons has a non-zero radius of convergence, whereas for polyominoes the radius of convergence is zero.

In section 2, we define the models. In section 3, we give the formula for $A_1(q, x)$, a generating function for $m = 1$ column-subconvex polyominoes, in which the variable q is conjugate to the area and x is conjugate to the number of columns of the polyomino. A truncated version of the proof is given in section 4. In section 5, we discuss the asymptotic behaviour of $A_1(q, x)$, and give a tight lower bound on the growth constant, as well as a very accurate estimate. In section 6, we give the formula for $G(q, w)$, a generating function for simple-2-column polyominoes on the square lattice, where once again the variable q is conjugate to the area, while w is conjugate to the number of two-component columns of the polyomino. For this model also we give an outline of the proof, which can be found in section 7. In section 8, we discuss the asymptotic behaviour of $G(q, w)$, including accurate numerical estimates and bounds. In section 9 we conclude, outlining further work prompted by our results.

Note that our solutions essentially give detailed information only about the area generating function. The additional parameters x and w count columns of a certain type. Thus, we cannot give perimeter–area phase diagrams which are relevant to the description of vesicle collapse. Indeed, as we show in section 9, the perimeter generating function has zero radius of convergence (as is the usual case for polyominoes), which precludes such a phase diagram in its usual form.

2. Definitions of the models

There are three regular tilings of the Euclidean plane, namely the triangular tiling, the square tiling and the hexagonal tiling. We adopt the convention that every square or hexagonal tile has two horizontal edges. In a regular tiling, a tile is often referred to as a *cell*. A plane figure P is a *polyomino* if P is a union of finitely many cells and the interior of P is connected. Observe that, if a union of *hexagonal* cells is connected, then it possesses a connected interior as well, as a connected union of hexagonal tiles must be connected through shared edges. Topologically, a connected union of square cells may be connected only at a shared vertex. Such unions are forbidden by the definition of polyominoes however.

Let P and Q be two polyominoes. We consider P and Q to be equal if and only if there exists a translation f such that $f(P) = Q$.

Given a polyomino P, it is useful to partition the cells of P according to their horizontal projection. Each block of that partition is a *column* of P. Note that a column of a polyomino is not necessarily a connected set. On the other hand, it may happen that every column of a polyomino P is a connected set. In this case, the polyomino P is a *column-convex polygon*. See figure 1.

By a *2-column polyomino*, we mean a polyomino in which columns with three or more connected components are not allowed. Thus, each column of a 2-column polyomino has either one or two connected components.

A polyomino P is a *level m column-subconvex polyomino* if the following holds.

- P is a 2-column polyomino,
- if a column of P has two connected components, then the gap between the components consists of at most m cells.

See figures 2 and 4*a*.

A *simple-2-column polyomino* is such a 2-column polyomino in which consecutive two-component columns are not allowed. If c is a column of a simple-2-column polyomino, and c is a (left or right) neighbour of a two-component column, then c must be a one-

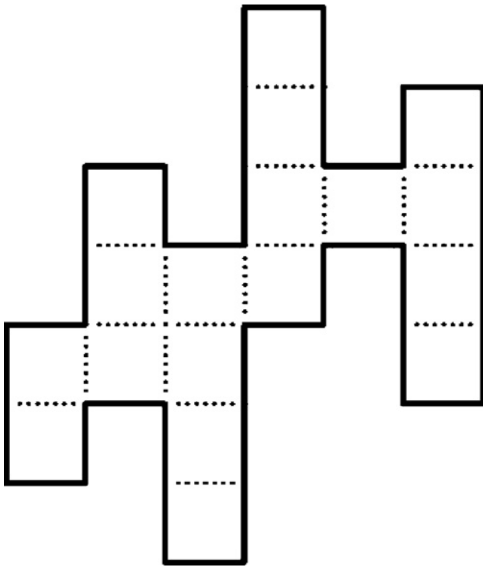

Figure 1. A column-convex polygon.

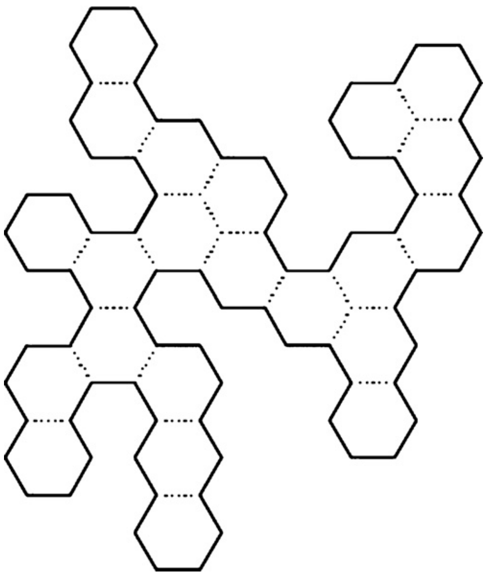

Figure 2. A level one column-subconvex polyomino.

component column. See figures 3 and 4*b*. Observe that, in a simple-2-column polyomino, a two-component column can have a hole of any size.

3. The area generating function for level one column-subconvex polyominoes with hexagonal cells

If a polyomino P is made up of n cells, we say that the *area* of P is n. Let $\mathcal{T}$ denote the set of all level one column-subconvex polyominoes with hexagonal cells. In theorem 1 below, we

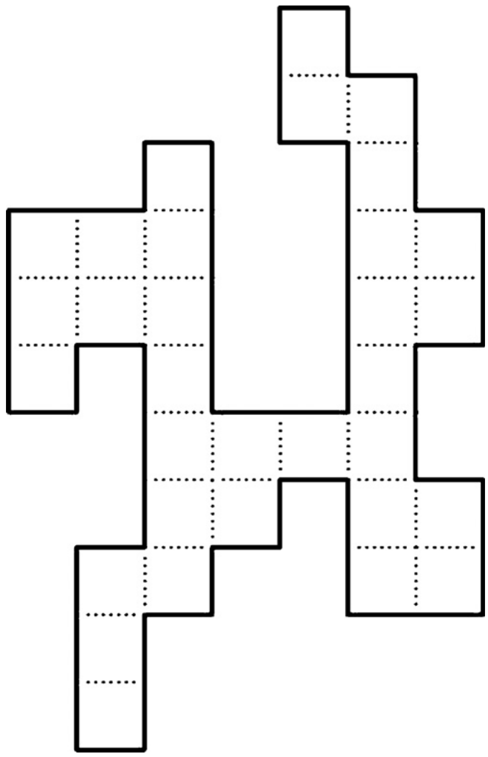

Figure 3. A simple-2-column polyomino.

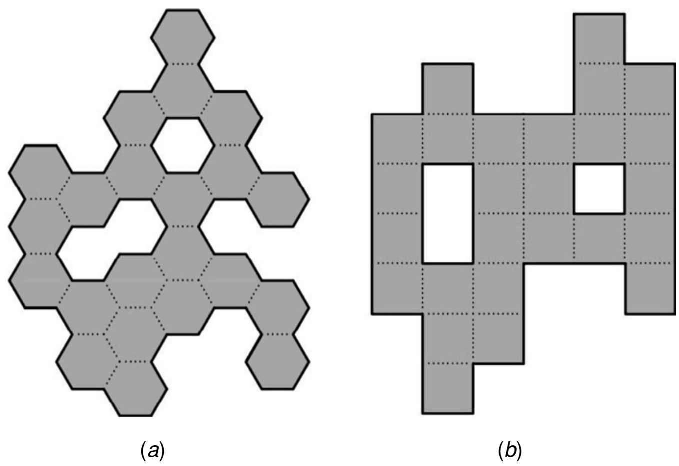

Figure 4. (a) Level one column-subconvex polyominoes can have internal holes. (b) The same holds for simple-2-column polyominoes.

state a formula for the generating function

$$A_1(q, x) = \sum_{P \in \mathcal{T}} q^{\text{area of } P} \cdot x^{\text{number of columns of } P}.$$

Theorem 1. *The generating function $A_1(q, x)$ is given by*

$$A_1(q, x) = \frac{\sum_{n=1}^{3} \mathrm{num_n}}{\sum_{n=1}^{6} \mathrm{den_n}}, \tag{1}$$

where

$$\begin{aligned}
\mathrm{num}_1 &= (q - 8q^2 + 28q^3 - 56q^4 + 70q^5 - 56q^6 + 28q^7 - 8q^8 + q^9)x \\
&\quad + (-4q^3 + 24q^4 - 58q^5 + 72q^6 - 48q^7 + 16q^8 - 2q^9)x^2 \\
&\quad + (5q^5 - 12q^6 + 12q^7 - 6q^8 + q^9)x^3, \\
\mathrm{num}_2 &= [(-q + 5q^2 - 9q^3 + 5q^4 + 5q^5 - 9q^6 + 5q^7 - q^8)x \\
&\quad + (-4q^3 + 18q^4 - 32q^5 + 28q^6 - 12q^7 + 2q^8)x^2 \\
&\quad + (5q^5 - 7q^6 + 5q^7 - q^8)x^3]\beta, \\
\mathrm{num}_3 &= (-2q^4 + 8q^5 - 12q^6 + 8q^7 - 2q^8)x^2\delta, \\
\mathrm{den}_1 &= 1 - 9q + 36q^2 - 84q^3 + 126q^4 - 126q^5 + 84q^6 - 36q^7 + 9q^8 - q^9 \\
&\quad + (-2q + 10q^2 - 16q^3 - 2q^4 + 40q^5 - 58q^6 + 40q^7 - 14q^8 + 2q^9)x \\
&\quad + (7q^3 - 36q^4 + 69q^5 - 60q^6 + 21q^7 - q^9)x^2 \\
&\quad + (-10q^5 + 10q^6 - 6q^7 + 2q^8)x^3, \\
\mathrm{den}_2 &= [(2q^2 - 12q^3 + 30q^4 - 40q^5 + 30q^6 - 12q^7 + 2q^8)x \\
&\quad + (4q^3 - 22q^4 + 46q^5 - 46q^6 + 22q^7 - 4q^8)x^2 \\
&\quad + (-10q^5 + 10q^6 - 6q^7 + 2q^8)x^3]\alpha, \\
\mathrm{den}_3 &= [-1 + 8q - 28q^2 + 56q^3 - 70q^4 + 56q^5 - 28q^6 + 8q^7 - q^8 \\
&\quad + (2q - 6q^2 - 2q^3 + 30q^4 - 50q^5 + 38q^6 - 14q^7 + 2q^8)x \\
&\quad + (13q^3 - 41q^4 + 48q^5 - 26q^6 + 7q^7 - q^8)x^2]\beta, \\
\mathrm{den}_4 &= [(2q^4 - 8q^5 + 12q^6 - 8q^7 + 2q^8)x^2 + (-4q^6 + 8q^7 - 4q^8)x^3]\gamma, \\
\mathrm{den}_5 &= [(6q^4 - 22q^5 + 30q^6 - 18q^7 + 4q^8)x^2 + (4q^6 - 4q^7)x^3]\delta, \\
\mathrm{den}_6 &= [(2q^4 - 6q^5 + 6q^6 - 2q^7)x^2 + (4q^6 - 4q^7)x^3](\alpha\delta - \beta\gamma), \\
\alpha &= \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}}}{(1-q)^i \left[\prod_{k=1}^{i}(1-q^{k+1})\right]^2}, \\
\beta &= \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}}}{(1-q)^i \left[\prod_{k=1}^{i-1}(1-q^{k+1})\right]^2 (1-q^{i+1})}, \\
\gamma &= \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}} \left(\frac{\mathrm{i}}{q} + 2\sum_{j=1}^{i} \frac{q^j}{1-q^{j+1}}\right)}{(1-q)^i \left[\prod_{k=1}^{i}(1-q^{k+1})\right]^2}, \\
\delta &= \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}} \left(\frac{\mathrm{i}}{q} + 2\sum_{j=1}^{i-1} \frac{q^j}{1-q^{j+1}} + \frac{q^i}{1-q^{i+1}}\right)}{(1-q)^i \left[\prod_{k=1}^{i-1}(1-q^{k+1})\right]^2 (1-q^{i+1})}.
\end{aligned}$$

4. Proof of theorem 1

Recall that $\mathcal{T}$ denotes the set of all level one column-subconvex polyominoes.

When we build a column-subconvex polyomino from left to right, adding one column at a time, the intermediate figures need not all be polyominoes, and therefore need not all be elements of $\mathcal{T}$. We say that a figure P is an *incomplete level one column-subconvex polyomino* if P itself is not an element of $\mathcal{T}$, but P is a 'left factor' of an element of $\mathcal{T}$. Note that, if P is an incomplete level one column-subconvex polyomino, then the last (i.e. the rightmost) column of P necessarily has a hole.

Let $\mathcal{U}$ denote the set of all incomplete level one column-subconvex polyominoes.

Let P be an element of $\mathcal{T} \cup \mathcal{U}$ and let P have at least two columns. Then we define

- the *body* of P to be all of P, except the rightmost column of P,
- the *lower pivot cell* of P to be the lower right neighbour of the lowest cell of the second last column of P,
- the *upper pivot cell* of P to be the upper right neighbour of the highest cell of the second last column of P.

We shall deal with the following generating functions:

$$A(q,x,t) = \sum_{P\in\mathcal{T}} q^{\text{area of } P} \cdot x^{\text{no. of columns of } P} \cdot t^{\text{the height of the last column of } P},$$

$$A_1 = A(q,x,1), \qquad B_1 = \frac{\partial A}{\partial t}(q,x,1),$$

$$C(q,x,u,v) = \sum_{P\in\mathcal{U}} q^{\text{area of } P} \cdot x^{\text{number of columns of } P} \cdot u^{\substack{\text{the height of the upper component}\\ \text{of the last column of } P}} \cdot v^{\substack{\text{the height of the lower component}\\ \text{of the last column of } P}},$$

$$D(u) = C(q,x,u,1), \qquad E(v) = C(q,x,1,v), \qquad C_1 = C(q,x,1,1).$$

In the above definitions, by the height of a holed column we mean the height of the upper component plus the height of the lower component plus one. (One is the height of the hole.)

Now we are going to partition the set $\mathcal{T}$ into six subsets: $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$, $\mathcal{T}_\gamma$, $\mathcal{T}_\delta$, $\mathcal{T}_\epsilon$ and $\mathcal{T}_\zeta$. The parts of the series A that come from the sets $\mathcal{T}_\alpha, \ldots, \mathcal{T}_\zeta$ will be denoted $A_\alpha, \ldots, A_\zeta$, respectively.

By $\mathcal{T}_\alpha$ we denote the set of level one column-subconvex polyominoes which have only one column. We have $A_\alpha = \frac{xqt}{1-qt}$.

By $\mathcal{T}_\beta$ we denote the set of all $P \in \mathcal{T}\backslash\mathcal{T}_\alpha$ which have the following properties: the body of P lies in $\mathcal{T}$, the last column of P has no hole, and the lower pivot cell of P is contained in P. We have $A_\beta = \frac{xqt}{(1-qt)^2} \cdot A_1$.

By $\mathcal{T}_\gamma$ we denote the set of all $P \in \mathcal{T}\backslash\mathcal{T}_\alpha$ which have the following properties: the body of P lies in $\mathcal{T}$, the last column of P has no hole, and the lower pivot cell of P is not contained in P. We have $A_\gamma = \frac{xqt}{1-qt} \cdot B_1$.

By $\mathcal{T}_\delta$ we denote the set of all $P \in \mathcal{T}\backslash\mathcal{T}_\alpha$ which have the following properties: the body of P lies in $\mathcal{T}$ and the last column of P has a hole. We have $A_\delta = \frac{xq^2t^3}{(1-qt)^2} \cdot (B_1 - A_1)$.

By $\mathcal{T}_\epsilon$ we denote the set of all $P \in \mathcal{T}\backslash\mathcal{T}_\alpha$ which have the following properties: the body of P lies in $\mathcal{U}$ and the last column of P has no hole. We have $A_\epsilon = \frac{xq^2t^2}{(1-qt)^2} \cdot C_1$.

By $\mathcal{T}_\zeta$ we denote the set of all $P \in \mathcal{T}\backslash\mathcal{T}_\alpha$ which have the following properties: the body of P lies in $\mathcal{U}$ and the last column of P has a hole. We have $A_\zeta = \frac{2xq^3t^4}{(1-qt)^3} \cdot C_1 - \frac{2xq^2t^3}{(1-qt)^3} \cdot D(qt)$.

Inserting the expressions for $A_\alpha, \ldots, A_\zeta$ into the equation $A = A_\alpha + \cdots + A_\zeta$, we obtain

$$\begin{aligned} A = \frac{xqt}{1-qt} + \frac{xqt}{(1-qt)^2} \cdot A_1 + \frac{xqt}{1-qt} \cdot B_1 + \frac{xq^2t^3}{(1-qt)^2} \cdot (B_1 - A_1) \\ + \frac{xq^2t^2}{(1-qt)^2} \cdot C_1 + \frac{2xq^3t^4}{(1-qt)^3} \cdot C_1 - \frac{2xq^2t^3}{(1-qt)^3} \cdot D(qt). \end{aligned} \tag{2}$$

Similarly, we partition the set $\mathcal{U}$ into five subsets: $\mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_\gamma, \mathcal{U}_\delta$ and $\mathcal{U}_\epsilon$. The parts of the series C that come from the sets $\mathcal{U}_\alpha, \ldots, \mathcal{U}_\epsilon$ are denoted as $C_\alpha, \ldots, C_\epsilon$, respectively.

By $\mathcal{U}_\alpha$ we denote the set of incomplete level one column-subconvex polyominoes which have only one column. We have $C_\alpha = \frac{xq^2uv}{(1-qu)(1-qv)}$.

By $\mathcal{U}_\beta$ we denote the set of all $P \in \mathcal{U}\backslash\mathcal{U}_\alpha$ which have the following properties: the body of P lies in $\mathcal{T}$, and the hole of the last column of P coincides either with the lower pivot cell of P or with the upper pivot cell of P. We have $C_\beta = \frac{2xq^2uv}{(1-qu)(1-qv)} \cdot A_1$.

By $\mathcal{U}_\gamma$ we denote the set of all $P \in \mathcal{U}\backslash\mathcal{U}_\alpha$ which have the following properties: the body of P lies in $\mathcal{T}$, and the hole of the last column of P lies either below the lower pivot cell of P or above the upper pivot cell of P. We have $C_\gamma = \frac{xq^2uv}{(1-qu)^2(1-qv)} \cdot A_1 + \frac{xq^2uv}{(1-qu)(1-qv)^2} \cdot A_1$.

By $\mathcal{U}_\delta$ we denote the set of all $P \in \mathcal{U}\backslash\mathcal{U}_\alpha$ which have the following properties: the body of P lies in $\mathcal{U}$, and the hole of the last column of P touches the hole of the second last column of P. We have $C_\delta = \frac{2xq^2uv}{(1-qu)(1-qv)} \cdot C_1$.

By $\mathcal{U}_\epsilon$ we denote the set of all $P \in \mathcal{U}\backslash\mathcal{U}_\alpha$ which have the following properties: the body of P lies in $\mathcal{U}$, and the hole of the last column of P does not touch the hole of the second last column of P. We have $C_\epsilon = \frac{xq^2uv}{(1-qu)(1-qv)^2} \cdot D(qv) + \frac{xq^2uv}{(1-qu)^2(1-qv)} \cdot D(qu)$.

Inserting the expressions for $C_\alpha, \ldots, C_\epsilon$ into the equation $C = C_\alpha + \cdots + C_\epsilon$, we obtain a functional equation for C. For our purposes, it will be enough to state the case $v = 1$ of that functional equation. With the notation

$$F = 1 + \frac{3-2q}{1-q} \cdot A_1 + 2C_1 + \frac{1}{1-q} \cdot D(q), \tag{3}$$

the case $v = 1$ of the functional equation for C reads

$$D(u) = \frac{xq^2u}{(1-q)(1-qu)^2} \cdot A_1 + \frac{xq^2u}{(1-q)(1-qu)} \cdot F + \frac{xq^2u}{(1-q)(1-qu)^2} \cdot D(qu). \tag{4}$$

The iteration of (4) produces

$$D(u) = \left\{ \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+3)}{2}} u^i}{(1-q)^i \cdot \left[\prod_{k=1}^{i}(1-q^k u)\right]^2} \right\} \cdot A_1 + \left\{ \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+3)}{2}} u^i}{(1-q)^i \cdot \left[\prod_{k=1}^{i-1}(1-q^k u)\right]^2 \cdot (1-q^i u)} \right\} \cdot F. \tag{5}$$

Then we set up a system of six linear equations in six unknowns: $A_1, B_1, C_1, D(q), D'(q)$ and F. One of the six equations is (3), and the other five are obtained as follows:

- by setting $t = 1$ in (2),
- by differentiating (2) with respect to t and then setting $t = 1$,
- by setting $u = 1$ in (4),
- by setting $u = q$ in (5),
- by differentiating (5) with respect to u and then setting $u = q$.

Once the linear system is solved, the proof of the theorem is complete.

5. The asymptotic analysis of $A_1(q, x)$

We write $[q^n]f$ to denote the coefficient of q^n in a power series $f = f(q)$.

From the solution above for $A_1(q, x)$, it is a straightforward matter to generate many hundreds of terms of the series $A_1(q, 1)$, corresponding to the area generating function of level one column-subconvex polyominoes. We have $A_1(q, 1) = q + 3q^2 + 11q^3 + 44q^4 + 184q^5 + 786q^6 + 3391q^7 + 14\,683q^8 + 63\,619q^9 + 275\,506q^{10} + 1192\,134q^{11} + 5154\,794q^{12} + 22\,278\,047q^{13} + 96\,250\,859q^{14} + \ldots$. The solution is too complicated to permit an analytic analysis of the asymptotics, so we resort to numerical methods. Fortunately, in this instance our methods are able to achieve almost any required accuracy.

One of the simplest things to try is to look at the ratio of successive terms. In the presence of an algebraic singularity, of the form $F(x) = \sum a_n x^n \sim A(1 - \mu x)^{-\gamma}$, one has

$$r_n = a_n/a_{n-1} = \mu[1 + (\gamma - 1)/n + \mathrm{o}(1/n)].$$

Depending on the nature of the singularity, the correction term $\mathrm{o}(1/n)$ can usually be made considerably sharper.

In the case of $A_1(q, 1)$, the ratios of successive terms are rapidly convergent, enabling us to estimate that the dominant singularity is the reciprocal of $\mu = 4.319\,139\,243\,729\,788\,226\,294\,125\,186\,813\,818\,984\,941\,600\,81$. The asymptotics are given by

$$[q^n]A_1(q, 1) = \lambda\mu^n + \mathrm{o}(\rho^{-n})$$

for any $1 < \rho < \rho_c$, where μ is given above and $\lambda = 0.122\,428\,100\,456\,122\,243\,205\,023\,911\,505\,973\,633\,306\,171\,383\ldots$ where we are confident that our estimates of μ and λ are correct to all quoted digits. We have been unable to find a convincing representation of μ in terms of the solution of any polynomial of degree less than 20. We also consider it likely that λ is a rational function of μ, but have not been able to identify it.

Simple concatenation arguments, first used by Klarner [18], enable one to prove that $\mu = \lim_{n\to\infty}[q^n]^{1/n} = \sup_n[q^n]^{1/n}$. In this way, making use of $[q^{250}]$, we find $\mu > 4.283\,00\ldots$, which is less than 1% below the best numerical estimate. Unfortunately, finding upper bounds is much more difficult.

With the singularity being a simple pole, subdominant terms are exponentially small. We can estimate the location of the first such singularity by the method of differential approximants [20] and find a conjugate pair at $q = q^* = 0.399\,878\mathrm{e}^{\pm \mathrm{i}\pi/9.4864}$. Thus, ρ_c defined above is given, approximately, by $0.399\,878 \times 4.3191 \approx 1.727$. Evidence of the phase factor can be seen by calculating a 'correction series', with coefficients given by $[q^n]A_1(q, 1) - \lambda\mu^n$. These coefficients have a periodicity in their sign pattern of about nine terms, corresponding to a phase factor close to $\mathrm{e}^{\pm \mathrm{i}\pi/9}$, exactly as found.

We can also write $A_1(q, x)$ as $\sum_{n>0} A_1^{(n)}(q)x^n$, where

$$A_1^{(1)}(q) = \frac{q}{1-q},$$

$$A_1^{(2)}(q) = \frac{2q^2(1-q+q^3)}{(1-q)^5(1+q)},$$

$$A_1^{(3)}(q) = \frac{q^3(4+q+\cdots-2q^7-q^8)}{(1-q)^8(1+q)^2(1+q+q^2)},$$

$$A_1^{(4)}(q) = \frac{2q^4(4+6q+\cdots-4q^{12}-2q^{13})}{(1-q)^{11}(1+q)^3(1+q+q^2)^2(1+q^2)}.$$

From this structure, we note that $A_1^{(n)}(q)$ is a rational function with denominators given by powers of cyclotomic polynomials of steadily increasing degree. (Indeed, the increases are very systematic, so that one could readily conjecture the pattern). If, as seems likely, this pattern persists, the zeros on the unit circle in the complex q plane will become dense. Such a function cannot be differentiably finite in x [4]. Admittedly, this is a plausibility argument, rather than a proof, that $A_1(q, 1)$ is not D-finite.

6. The area generating function for simple-2-column polyominoes with square cells

Let $\mathcal{R}$ denote the set of all simple-2-column polyominoes with square cells.

In theorem 2 below, we state a formula for the generating function

$$G(q, w) = \sum_{P \in \mathcal{R}} q^{\text{area of } P} \cdot w^{\text{number of two-component columns of } P}.$$

Theorem 2. *The generating function $G(q, w)$ is given by*

$$G(q, w) = \frac{\mathrm{NUM}}{\mathrm{DEN}}, \tag{6}$$

where

$$\begin{aligned}
\mathrm{NUM} = {} & (1-q)^4(\tilde{\alpha} + \tilde{\gamma} + 2\tilde{\alpha}\tilde{\eta} - 2\tilde{\gamma}\tilde{\epsilon}) + q^2 w(1-q)^2(\tilde{\iota} + \tilde{\lambda} - \tilde{\alpha}\tilde{\kappa} - \tilde{\alpha}\tilde{\mu} \\
& + \tilde{\beta}\tilde{\iota} + \tilde{\beta}\tilde{\lambda} - \tilde{\gamma}\tilde{\kappa} - \tilde{\gamma}\tilde{\mu} + \tilde{\delta}\tilde{\iota} + \tilde{\delta}\tilde{\lambda} - 2\tilde{\epsilon}\tilde{\lambda} + 2\tilde{\eta}\tilde{\iota} + 2\tilde{\alpha}\tilde{\zeta}\tilde{\lambda} - 2\tilde{\alpha}\tilde{\eta}\tilde{\kappa} \\
& - 2\tilde{\alpha}\tilde{\eta}\tilde{\mu} + 2\tilde{\alpha}\tilde{\theta}\tilde{\lambda} - 2\tilde{\beta}\tilde{\epsilon}\tilde{\lambda} + 2\tilde{\beta}\tilde{\eta}\tilde{\iota} + 2\tilde{\gamma}\tilde{\epsilon}\tilde{\kappa} + 2\tilde{\gamma}\tilde{\epsilon}\tilde{\mu} - 2\tilde{\gamma}\tilde{\zeta}\tilde{\iota} - 2\tilde{\gamma}\tilde{\theta}\tilde{\iota} \\
& - 2\tilde{\delta}\tilde{\epsilon}\tilde{\lambda} + 2\tilde{\delta}\tilde{\eta}\tilde{\iota}) + 2q^2 w(1-q^2)(\tilde{\alpha}\tilde{\lambda} - \tilde{\gamma}\tilde{\iota}), \\
\mathrm{DEN} = {} & (1-q)^4(1 - \tilde{\beta} + \tilde{\delta} - \tilde{\epsilon} + \tilde{\eta} - \tilde{\alpha}\tilde{\zeta} + \tilde{\alpha}\tilde{\theta} + \tilde{\beta}\tilde{\epsilon} - \tilde{\beta}\tilde{\eta} + \tilde{\gamma}\tilde{\zeta} - \tilde{\gamma}\tilde{\theta} \\
& - \tilde{\delta}\tilde{\epsilon} + \tilde{\delta}\tilde{\eta}) - 2(1-q)^3(\tilde{\gamma} + \tilde{\alpha}\tilde{\eta} - \tilde{\gamma}\tilde{\epsilon}) \\
& - 2q^2 w(1-q)^2(\tilde{\kappa} - \tilde{\beta}\tilde{\mu} + \tilde{\delta}\tilde{\kappa} - \tilde{\epsilon}\tilde{\kappa} + \tilde{\zeta}\tilde{\iota} - \tilde{\zeta}\tilde{\lambda} + \tilde{\eta}\tilde{\kappa} - \tilde{\alpha}\tilde{\zeta}\tilde{\mu} + \tilde{\alpha}\tilde{\theta}\tilde{\kappa} \\
& + \tilde{\beta}\tilde{\epsilon}\tilde{\mu} - \tilde{\beta}\tilde{\eta}\tilde{\mu} - \tilde{\beta}\tilde{\theta}\tilde{\iota} + \tilde{\beta}\tilde{\theta}\tilde{\lambda} + \tilde{\gamma}\tilde{\zeta}\tilde{\mu} - \tilde{\gamma}\tilde{\theta}\tilde{\kappa} - \tilde{\delta}\tilde{\epsilon}\tilde{\kappa} + \tilde{\delta}\tilde{\zeta}\tilde{\iota} - \tilde{\delta}\tilde{\zeta}\tilde{\lambda} + \tilde{\delta}\tilde{\eta}\tilde{\kappa}) \\
& - 4q^2 w(1-q)(\tilde{\beta}\tilde{\lambda} - \tilde{\gamma}\tilde{\kappa} + \tilde{\alpha}\tilde{\zeta}\tilde{\lambda} - \tilde{\alpha}\tilde{\eta}\tilde{\kappa} - \tilde{\beta}\tilde{\epsilon}\tilde{\lambda} + \tilde{\beta}\tilde{\eta}\tilde{\iota} + \tilde{\gamma}\tilde{\epsilon}\tilde{\kappa} - \tilde{\gamma}\tilde{\zeta}\tilde{\iota}) \\
& - 2q^3 w(1-q)(\tilde{\iota} + \tilde{\alpha}\tilde{\kappa} - \tilde{\alpha}\tilde{\mu} - \tilde{\beta}\tilde{\iota} + \tilde{\delta}\tilde{\iota} - \tilde{\epsilon}\tilde{\lambda} + \tilde{\eta}\tilde{\iota} - \tilde{\alpha}\tilde{\zeta}\tilde{\lambda} + \tilde{\alpha}\tilde{\eta}\tilde{\kappa} \\
& - \tilde{\alpha}\tilde{\eta}\tilde{\mu} + \tilde{\alpha}\tilde{\theta}\tilde{\lambda} + \tilde{\beta}\tilde{\epsilon}\tilde{\lambda} - \tilde{\beta}\tilde{\eta}\tilde{\iota} - \tilde{\gamma}\tilde{\epsilon}\tilde{\kappa} + \tilde{\gamma}\tilde{\epsilon}\tilde{\mu} + \tilde{\gamma}\tilde{\zeta}\tilde{\iota} - \tilde{\gamma}\tilde{\theta}\tilde{\iota} - \tilde{\delta}\tilde{\epsilon}\tilde{\lambda} + \tilde{\delta}\tilde{\eta}\tilde{\iota}) \\
& - 4q^3 w(\tilde{\alpha}\tilde{\lambda} - \tilde{\gamma}\tilde{\iota}),
\end{aligned}$$

$$\tilde{\beta} = \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{2}}},$$

$$\tilde{\gamma} = \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+4i} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{3}}},$$

$$\tilde{\zeta} = \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{2}}} \cdot \left(2i - 2 + 4 \cdot \sum_{k=1}^{i-1} \frac{q^k}{1-q^k} + \frac{\overline{2}q^i}{1-q^i}\right),$$

$$\tilde{\eta} = \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+4i} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{3}}}$$

$$\cdot \left(2i + 4 \cdot \sum_{k=1}^{i-1} \frac{q^k}{1-q^k} + \frac{\overline{3}q^i}{1-q^i}\right),$$

$$\tilde{\kappa} = \frac{1}{2} \cdot \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{2}}}$$

$$\cdot \left[\left(2i - 2 + 4 \cdot \sum_{k=1}^{i-1} \frac{q^k}{1-q^k} + \frac{\overline{2}q^i}{1-q^i}\right)^2 \right.$$

$$\left. - 2i + 2 + 4 \cdot \sum_{k=1}^{i-1} \frac{q^{2k}}{(1-q^k)^2} + \frac{\overline{2}q^{2i}}{(1-q^i)^2}\right],$$

$$\tilde{\lambda} = \frac{1}{2} \cdot \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+4i} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{3}}}$$

$$\cdot \left[\left(2i + 4 \cdot \sum_{k=1}^{i-1} \frac{q^k}{1-q^k} + \frac{\overline{3}q^i}{1-q^i}\right)^2 \right.$$

$$\left. - 2i + 4 \cdot \sum_{k=1}^{i-1} \frac{q^{2k}}{(1-q^k)^2} + \frac{\overline{3}q^{2i}}{(1-q^i)^2}\right].$$

In the above formulae, it will be noticed that (a) some of the numbers have an overline, and (b) no result is given for $\tilde{\alpha}$, $\tilde{\delta}$, $\tilde{\epsilon}$, $\tilde{\theta}$, $\tilde{\iota}$ and $\tilde{\mu}$. This is both to save space and to highlight the close similarity between certain quantities. For all the quantities defined above, the overlines may be ignored. To obtain the formula for $\tilde{\alpha}$ from the formula for $\tilde{\beta}$, replace $\overline{2}$ by 1. To obtain the formula for $\tilde{\delta}$ from the formula for $\tilde{\gamma}$, replace $\overline{3}$ by 4 and change $(-3)^{i-1}$ to $(-3)^i$. To obtain the formula for $\tilde{\epsilon}$ from the formula for $\tilde{\zeta}$, and also to obtain the formula for $\tilde{\iota}$ from the formula for $\tilde{\kappa}$, replace each of the $\overline{2}$'s by 1. To obtain the formula for $\tilde{\theta}$ from the formula for $\tilde{\eta}$, and also to obtain the formula for $\tilde{\mu}$ from the formula for $\tilde{\lambda}$, replace each of the $\overline{3}$'s by 4 and change $(-3)^{i-1}$ into $(-3)^i$.

7. Proof of theorem 2

Let P be a simple-2-column polyomino and let P have at least two one-component columns. Then we define the *lower pivot cell* of P to be the cell which is the right neighbour of the bottom cell of the second-last (i.e. second-rightmost) among the one-component columns of P. We also define the *upper pivot cell* of P to be the right neighbour of the top cell of the second-last among the one-component columns of P.

Let P be a simple-2-column polyomino and let P have at least one two-component column. Then we define the *lower inner pivot cell* of P to be the right neighbour of the top cell of the lower component of the last among the two-component columns of P. We also define the *upper inner pivot cell* of P to be the right neighbour of the bottom cell of the upper component of the last among the two-component columns of P.

Observe that the lower pivot cell of a simple-2-column polyomino P is not necessarily contained in P. The same holds for the upper pivot cell, the lower inner pivot cell and the upper inner pivot cell of P.

Let $\mathcal{S}$ denote the set of those simple-2-column polyominoes whose last (i.e. rightmost) column is a one-component column. It is convenient to first compute a generating function for the set $\mathcal{S}$, and thence a generating function for all simple-2-column polyominoes. So, let

$$H(q,t,w) = \sum_{P\in\mathcal{S}} q^{\text{area of } P} \cdot t^{\substack{\text{the height of the}\\ \text{last column of } P}} \cdot w^{\substack{\text{the number of}\\ \text{two-component columns of } P}}.$$

Next, we define three generating functions in two variables, q and w: let

$$H_1 = H(q,1,w),\ I_1 = \left\{\frac{\partial\left[\frac{H(q,t,w)}{t}\right]}{\partial t}\right\}_{with\, t=1} \quad\text{and}\quad J_1 = \frac{1}{2}\cdot\left\{\frac{\partial^2\left[\frac{H(q,t,w)}{t}\right]}{\partial t^2}\right\}_{with\, t=1}.$$

The generating functions $G(q,w)$ and $H(q,t,w)$ are related by

$$G(q,w) = H_1 + \frac{q^2 w}{(1-q)^2}\cdot J_1. \tag{7}$$

Henceforth, the notation $H(q,t,w)$ will be abbreviated as $H(t)$.

In order to obtain a functional equation for the generating function $H(t)$, we are going to suitably partition the set $\mathcal{S}$. The blocks of the partition will be denoted as $\mathcal{S}_\alpha, \mathcal{S}_\beta, \ldots, \mathcal{S}_\mu$, and the parts of $H(t)$ coming from these blocks will be denoted as $H_\alpha(t), H_\beta(t), \ldots, H_\mu(t)$, respectively.

First, we define $\mathcal{S}_\alpha$ to be the set of those $P \in \mathcal{S}$ which have no other one-component column than the last column. We have $H_\alpha(t) = \frac{qt}{1-qt} + \frac{q^5t^3w}{(1-q)^2(1-qt)^3}$.

Let $\mathcal{S}_\beta$ be the set of those $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ which have the following two properties: the second-last column is a one-component column, and the last column contains the lower pivot cell of P. We have $H_\beta(t) = \frac{qt}{(1-qt)^2}\cdot H_1$.

Let $\mathcal{S}_\gamma$ be the set of those $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ which have the following two properties: the second-last column is a one-component column, and the last column does not contain the lower pivot cell of P. We have $H_\gamma(t) = \frac{qt}{1-qt}\cdot I_1$.

Thus, $\mathcal{S}_\beta \cup \mathcal{S}_\gamma$ is the set of those $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ whose second-last column is a one-component column.

Let $\mathcal{S}_\delta$ be the set of those $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ which have the following three properties: the second-last column is a two-component column and the third-last column is (necessarily) a one-component column, the lower component of the second-last column and the third-last column have no edge in common, the lower pivot cell of P is contained in the upper component of the second-last column. We have $H_\delta(t) = \frac{q^5t^3w}{(1-q)^3(1-qt)^3}\cdot H_1$.

Let $\mathcal{S}_\epsilon$ be the set of $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ having the following three properties: the second-last column is a two-component column and the third-last column is a one-component column, the lower component of the second-last column and the third-last column have no edge in common, and the lower pivot cell of P is contained in the hole of the second-last column. We have $H_\epsilon(t) = \frac{q^5t^3w}{(1-q)^2(1-qt)^4}\cdot H_1 - \frac{q^4t^2w}{(1-q)^2(1-qt)^4}\cdot H(qt)$.

The definition of $\mathcal{S}_\zeta$ is obtained from the definition of $\mathcal{S}_\delta$ by writing the word 'upper' where the definition of $\mathcal{S}_\delta$ says 'lower', and by writing the word 'lower' where the definition of $\mathcal{S}_\delta$ says 'upper'. We have $H_\zeta(t) = H_\delta(t)$.

The definition of $\mathcal{S}_\eta$ is obtained when the changes just described are made to the definition of $\mathcal{S}_\epsilon$ (instead of to the definition of $\mathcal{S}_\delta$). We have $H_\eta(t) = H_\epsilon(t)$.

Thus, $\mathcal{S}_\delta \cup \mathcal{S}_\epsilon \cup \mathcal{S}_\zeta \cup \mathcal{S}_\eta$ is the set of those $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ which, in addition to having a two-component second-last column, also have the property that $P\backslash(\text{the last column of } P)$ is not a polyomino.

Let $\mathcal{S}_\theta$ be the set of $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ which have the following three properties: the second-last column is a two-component column and the third-last column is a one-component column, each of the two components of the second-last column has at least one edge in common with the third-last column, and the last column also has at least one edge in common with each of the two components of the second-last column. We have $H_\theta(t) = \frac{q^5t^3w}{(1-q)^2(1-qt)^3} \cdot I_1 - \frac{q^5t^3w}{(1-q)^2(1-qt)^4} \cdot H_1 + \frac{q^4t^2w}{(1-q)^2(1-qt)^4} \cdot H(qt)$.

Let $\mathcal{S}_\iota$ be the set of $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ which have the following four properties.

- The second-last column is a two-component column and the third-last column is a one-component column,
- Each of the two components of the second-last column has at least one edge in common with the third-last column,
- The last column has at least one edge in common with the lower component of the second-last column, but does not have any edges in common with the upper component of the second-last column,
- The last column does not contain the lower inner pivot cell of P.

We have $H_\iota(t) = \frac{q^4tw}{(1-q)^3(1-qt)} \cdot J_1$.

Let $\mathcal{S}_\kappa$ be the set of $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ having the following four properties.

- The second-last column is a two-component column and the third-last column is a one-component column.
- Each of the two components of the second-last column has at least one edge in common with the third-last column.
- The last column has at least one edge in common with the lower component of the second-last column, but does not have any edges in common with the upper component of the second-last column.
- The last column contains the lower inner pivot cell of P.

We have

$$H_\kappa(t) = \frac{q^3tw}{(1-q)^2(1-qt)^2} \cdot J_1 - \frac{q^5t^3w}{(1-q)^2(1-qt)^3} \cdot I_1 + \frac{q^5t^3w}{(1-q)^2(1-qt)^4} \cdot H_1 - \frac{q^4t^2w}{(1-q)^2(1-qt)^4} \cdot H(qt).$$

The definition of $\mathcal{S}_\lambda$ is obtained from the definition of $\mathcal{S}_\iota$ by writing the word 'upper' where the definition of $\mathcal{S}_\iota$ says 'lower', and by writing the word 'lower' where the definition of $\mathcal{S}_\iota$ says 'upper'. We have $H_\lambda(t) = H_\iota(t)$.

The definition of $\mathcal{S}_\mu$ is obtained when the changes just described are made to the definition of $\mathcal{S}_\kappa$ (instead of to the definition of $\mathcal{S}_\iota$). We have $H_\mu(t) = H_\kappa(t)$.

Thus, $\mathcal{S}_\theta \cup \mathcal{S}_\iota \cup \mathcal{S}_\kappa \cup \mathcal{S}_\lambda \cup \mathcal{S}_\mu$ is the set of those $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ which, in addition to having a two-component second-last column, also have the property that $P\backslash$(the last column of P) is a polyomino. This means that $\mathcal{S}_\delta \cup \mathcal{S}_\epsilon \cup \ldots \cup \mathcal{S}_\mu$ is the set of all $P \in \mathcal{S}\backslash\mathcal{S}_\alpha$ whose second-last column is a two-component column.

The sets $\mathcal{S}_\alpha, \mathcal{S}_\beta, \ldots, \mathcal{S}_\mu$ form a partition of the set $\mathcal{S}$. We have $H(t) = H_\alpha(t) + H_\beta(t) + \cdots + H_\mu(t)$, and the expressions for $H_\alpha(t), H_\beta(t), \ldots, H_\mu(t)$ are given above. Putting these things together, we get a functional equation for $H(t)$. It is convenient to write that functional equation as

$$\begin{aligned} H(t) = {} & \frac{qt}{1-qt} \cdot \left[1 + I_1 + \frac{2q^3w}{(1-q)^3} \cdot J_1\right] + \frac{qt}{(1-qt)^2} \cdot \left[H_1 + \frac{2q^2w}{(1-q)^2} \cdot J_1\right] \\ & + \frac{q^5t^3w}{(1-q)^2(1-qt)^3} \cdot \left(1 + \frac{2}{1-q} \cdot H_1 - I_1\right) + \frac{3q^5t^3w}{(1-q)^2(1-qt)^4} \cdot H_1 \\ & - \frac{3q^4t^2w}{(1-q)^2(1-qt)^4} \cdot H(qt). \end{aligned} \tag{8}$$

We solved equation (8) by iteration, as is usually done in the upgraded Temperley method. The iteration ended in

$$
\begin{aligned}
H(t) = {} & \left\{ \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} t^{2i-1} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k t)\right]^4 \cdot (1-q^i t)} \right\} \cdot \left[1 + I_1 + \frac{2q^3 w}{(1-q)^3} \cdot J_1 \right] \\
& + \left\{ \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} t^{2i-1} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k t)\right]^4 \cdot (1-q^i t)^2} \right\} \cdot \left[H_1 + \frac{2q^2 w}{(1-q)^2} \cdot J_1 \right] \\
& + \left\{ \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+4i} t^{2i+1} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i-1}(1-q^k t)\right]^4 \cdot (1-q^i t)^3} \right\} \cdot \left(1 + \frac{2}{1-q} \cdot H_1 - I_1 \right) \\
& - \left\{ \sum_{i=1}^{\infty} \frac{(-3)^{i} q^{i^2+4i} t^{2i+1} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i}(1-q^k t)\right]^4} \right\} \cdot H_1 . \qquad (9)
\end{aligned}
$$

From equation (9), we got a system of three linear equations in three unknowns: H_1, I_1 and J_1. The first equation is just the case $t = 1$ of (9). The second equation is obtained by dividing (9) by t, differentiating with respect to t and then setting $t = 1$. The third equation is obtained by dividing (9) by t, differentiating twice with respect to t and then setting $t = 1$.

Once the linear system is solved, relation (7) tells us how to obtain the sought-after generating function G. (To solve the linear system, we made use of the computer algebra package *Maple*.)

7.1. A corollary to theorem 2

By setting $w = 0$, from theorem 2 we obtain the well-known result, discovered independently by Temperley [22] and Pólya [19].

Corollary 1. *The area generating function for column-convex polygons with square cells is given by*

$$G(q, 0) = \frac{q(1-q)^3}{1 - 5q + 7q^2 - 4q^3}.$$

8. The asymptotic analysis of $G(q, w)$

Our analysis of the $G(q, w)$ series parallels that given in section 5, as the singularities here are also simple poles. Note that $G(q, 0)$ is dominated by a simple pole at the smallest zero of $1 - 5q + 7q^2 - 4q^3$, which is at $q = q_c = 0.311\,957\,055\,278\ldots$. From the solution above for $G(q, w)$, it is again a straightforward matter to generate many hundreds of terms of the series $G(q, 1)$, corresponding to the area generating function of simple-2-column polyominoes. We have $G(q, 1) = q + 2q^2 + 6q^3 + 19q^4 + 63q^5 + 216q^6 + 758q^7 + 2693q^8 + 9608q^9 + 34\,269q^{10} + 121\,946q^{11} + 432\,701q^{12} + 1531\,246q^{13} + \ldots$. The solution is too complicated to permit an analytic analysis of the asymptotics, so we resort to numerical methods once again.

Writing $G(q, 1) = \sum b_n q^n$, we find $b_{50}/b_{49} = 3.522\,019\,842$, $b_{100}/b_{99} = 3.522\,019\,812\,881\,5885$, $b_{150}/b_{149} = 3.522\,019\,812\,881\,584\,830\,067\,67$, $b_{200}/b_{199} = 3.522\,019\,812\,881\,584\,830\,067\,520\,977\,156\,64$ and $b_{250}/b_{249} = 3.522\,019\,812\,881\,584\,830\,067\,520\,977\,156\,868\,436\,53$. It can be seen that each additional 50 terms add approximately 8 significant digits to the estimate of μ, the limiting value of the ratios. Thus, the ratios are approaching μ when extrapolated against $1/n$, with zero slope, corresponding to a simple pole singularity, as might have been expected by analogy with the behaviour of $G(q, 0)$.

We can now write the asymptotics much more precisely. We have that

$$[q^n]G(q, 1) = \lambda\mu^n + \mathrm{o}(\rho^{-n})$$

for any $1 < \rho < \rho_c$, where $\mu = 3.522\,019\,812\,881\,584\,830\,067\,520\,977\,156\,868\,436\,53\ldots$. We will shortly provide an estimate of ρ_c. To calculate the amplitude λ, we simply compute the sequence $[q^n]G(q, 1)/\mu^n$, which is also rapidly convergent, so that we may write $\lambda = 0.119\,442\,870\,404\,867\,084\,313\,264\,237\,052\,704\,329\,586\ldots$ where we are confident that our estimates of μ and λ are correct to all quoted digits. By analogy with some other solved polygon models [11] we hoped to identify μ as an algebraic number, but have been unable to find a convincing representation in terms of the solution of any polynomial of degree less than 20. We also consider it likely that λ is a rational function of μ, but have not been able to identify it.

As in section 5, we can get a tight lower bound from $\mu = \lim_{n\to\infty}[q^n]^{1/n} = \sup_n [q^n]^{1/n}$. Making use of $[q^{250}]$ we find $\mu > 3.492\,09\ldots$, which is less than 1% below the best numerical estimate.

With the singularity being a simple pole, subdominant terms are exponentially small. We can estimate the location of the first such singularity by the method of differential approximants [20] and find a conjugate pair of singularities at $q = q^* = 0.400\mathrm{e}^{\pm \mathrm{i}\pi/8.88}$. Thus, ρ_c defined above is given, approximately, by $0.400 \times 3.522 \approx 1.41$. Evidence of the phase factor can be seen by calculating a 'correction series', with coefficients given by $[q^n]G(q, 1) - \lambda\mu^n$. These coefficients have a periodicity in their sign pattern of about nine terms, corresponding to a phase factor close to $\mathrm{e}^{\pm\mathrm{i}\pi/9}$, exactly as found.

We can also write $G(q, w)$ as $\sum_n G_n(q)w^n$, where

$$G_0(q) = G(q, 0) = \frac{q(1-q)^3}{\Lambda},$$

with $\Lambda = 1 - 5q + 7q^2 - 4q^3$,

$$G_1(q) = \frac{q^5(2 + 3q^3 - 4q^4 + q^5 + 2q^6 - 7q^7 + 4q^8 + q^9)}{(1-q^2)^3\Lambda^2},$$

$$G_2(q) = \frac{q^7(1 + 2q + \cdots + 19q^{25} + q^{26})}{(1-q^2)^6(1-q^3)^3\Lambda^3},$$

$$G_3(q) = \frac{q^{12}(9 + 36q + \cdots + 32q^{41} + q^{42})}{(1-q^2)^6(1-q)^2(1-q^3)^4(1-q^4)^3\Lambda^4}.$$

From this structure, we can make several remarks. Firstly, note that each term in the expansion has a pole at the zero of Λ, whereas the sum of the terms has a pole closer to the origin at $q = 1/\mu$, as shown above. Secondly, note that $\Lambda^{n+1}G_n(q)$ is a rational function with denominators given by powers of cyclotomic polynomials of steadily increasing degree. If, as seems likely, this pattern persists, the zeros on the unit circle in the complex q plane will become dense. Such a function cannot be differentiably finite in w [4]. While this does not, in principle, exclude the possibility that $G(q, 1)$ could be D-finite, it would have to be a pathological function indeed that behaved in this way. Of course, pathological functions exist, so our argument is just that—a plausibility argument, and not a proof.

9. Further work

Our next goal will be to find the area generating function for simple-2-column polyominoes with hexagonal cells. That should not be difficult because we already have a method which works with square cells. It is usually possible to make such a method work when cells are hexagons as well.

Unlike the simple, rational expression given above for the area generating function of column-convex polygons, the area generating function of convex polygons is a sum of rational functions of q-series [2]—not unlike our solution for the area generating function of simple-2-column polyominoes, though not as complicated. For convex polygons, it is also possible to find the generating function by perimeter. This was first given in [5] and was later obtained independently in [13]. However, if one asks for the perimeter generating function of simple-2-column polyominoes, it turns out that this has zero radius of convergence. We show this by a very simple argument.

Consider a square of side $2n + 1$ sites. This clearly has perimeter $8n + 4$. Then construct a simple-2-column polyomino by placing a single square (of perimeter 4) in any of the square cells of the second column, except the top and bottom. This can be done in $2n - 1$ ways. Now repeat this for the fourth column, the sixth column, up to the $2n$th column. We have therefore placed n squares inside the large square, so the total perimeter of our object, which is a simple-2-column polyomino, is now $12n + 4$. The squares can be placed in $(2n - 1)^n$ ways. Thus, if p_{2n} denotes the number of simple-2-column polyominoes of perimeter $2n$, we have $p_{12n+4} \geqslant (2n-1)^n$. The large n limit of $\frac{1}{2n} \log p_{2n}$ diverges; hence, the radius of convergence is zero. While this does not mean that the perimeter generating function is uninteresting, it would be a whole new research project to study the nature of the singularity and its significance, and will not be discussed further in this paper.

In terms of possible extensions of this work, it is probably possible to compute the area generating function of *simple-2-column*$_2$ polyominoes. Here, by a simple-2-column$_2$ polyomino we mean a 2-column polyomino in which runs of two consecutive two-component columns are allowed, but it is forbidden for three consecutive columns to each have two connected components. One reason for doing this is that the growth factor μ is expected to be greater than that for simple-2-column polyominoes, and may set the benchmark in this regard. At present the situation is that for column-convex polygons the growth constant is $\mu = 3.205\,569\ldots$, while for simple-2-column polyominoes the growth constant is $\mu = 3.522\,0198\ldots$. For polyominoes the best lower bound [1] is $\mu \geqslant 3.980\,137$, which is quite close to the best estimate [20] $\mu \approx 4.062\,5696$. The polyomino model with a growth constant closest to the actual value for polyominoes is a directed model called *multi-directed polyominoes* [4] with a growth constant of $\mu \approx 3.58$. It would be interesting to compute the area of simple-2-column$_2$ polyominoes to see if they had a growth constant closer still to that for polyominoes.

As regards column-subconvex polyominoes, the above argument may be repeated *mutatis mutandis* to show that the perimeter generating function will also have zero radius of convergence. The growth constant for this model, when enumerated by area, is $\mu = 4.319\,139\ldots$, which may be compared to the best estimate for hexagonal polyominoes [20] of $\mu \approx 5.183\,1453$. The enumeration by area of the level 2 model is possible, but takes a lot of efforts. We did perform that enumeration, which will be published subsequently, and the formula for the area generating function of level two column-subconvex polyominoes is available in [24]. The growth constant is found to increase to $\mu = 4.509\,48\ldots$ in that case.

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